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SIMULTANEOUS TESTS FOR THE EQUALITY OF VARIANCES AGAINST CERTAIN ALTERNATIVES

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SIMULTANEOUS TESTS FOR THE EQUALITY OF VARIANCES AGAINST CERTAIN ALTERNATIVES[1]

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1. Introduction

In many situations, the experimenter is interested in testing which of the variances differ significantly from others when the overall hypothesis of equality of variances is rejected. The simultaneous test procedures play an important role in these situations. Hartley (1950) considered the problem of pairwise comparisons of variances simultaneously. Gnanadesikan (1959) considered the problem of comparing several variances simultaneously against a standard one. In the present paper, test procedures are proposed for comparing each variance simultaneously against the next one by using the union-intersection principle of Roy (1953).

2. Simultaneous Test Procedures

Consider k normal populations with means (known or unknown) $\mu_1, \mu_2, \ldots \mu_k$ and unknown variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Let s_i^2 be the sample estimate of σ_i^2 based on n_i degrees of freedom. In addition, let $\mathrm{H}: \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$. The total hypothesis H can be expressed as $\mathrm{H} = \bigcap_{i \neq j=1}^{k} \mathrm{H}_{ij}$, $\mathrm{H} = \bigcap_{i=1}^{k-1} \mathrm{H}_{ik}$ and $\mathrm{H} = \bigcap_{i=1}^{k-1} \mathrm{H}_{i,i+1}$ where $\mathrm{H}_{rs}: \sigma_r^2 = \sigma_s^2$. Now, let $\mathrm{A}_1 = \bigcup_{i \neq j=1}^{k} \mathrm{A}_{ij}$, $\mathrm{A}_2 = \bigcup_{i=1}^{k-1} \mathrm{A}_{ik}$ and $\mathrm{A}_3 = \bigcup_{i=1}^{k-1} \mathrm{A}_{i,i+1}$ where $\mathrm{A}_{rs}: \sigma_r^2 \neq \sigma_s^2$.

Hartley (1950) considered the problems of testing H against A_1 when the sample sizes are equal. This test is known as the F_{max} test. Ramachandran (1956) showed that Hartley's F_{max} test is unbiased. The tables which are useful in the application of this test procedure are given in David (1952).

Gnanadesikan (1959) considered the problems of testing H against A_2; his proof for the monotonicity of the power of the test is not correct. We now discuss a one-sided version of the test considered in Gnanadesikan (1959).

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A procedure for testing $H_{1k}, H_{2k}, \ldots, H_{k-1,k}$ and H simultaneously against the respective alternatives $A^{*}_{1k}, A^{*}_{2k}, \ldots, A^{*}_{k-1,k}$ and A^{*}_{2}, where $A^{*}_{2} = \bigcup_{i=1}^{k-1} A^{*}_{ik}$ and $A^{*}_{ik} : \sigma_i^2 > \sigma_k^2$ is discussed here.

According to this procedure, we accept or reject H_{ik} according as

$$F_{ik} \lesseqgtr c_{i\alpha}$$

where $c_{i\alpha}$'s are chosen such that

$$P[F_{ik} \leq c_{i\alpha}\,;\ i=1, 2, \ldots, k-1 | H] = (1-\alpha), \text{ and } F_{ij} = s_i^2/s_j^2. \tag{2.1}$$

The total hypothesis H is accepted if all the individual hypotheses $H_{1k}, H_{2k}, \ldots, H_{k-1,k}$ are accepted. This test procedure is similar to the test proposed by Ghosh (1955) for testing several hypotheses simultaneously under the ANOVA Model I. For practical purposes, we choose the critical values $c_{i\alpha}$ to be equal, and call them c_α. When $n_1 = n_2 = \ldots = n_k = n$, we can obtain the critical values c_α from the tables of Gupta (1963) for $\alpha = 0\cdot25, 0\cdot10, 0\cdot05, 0\cdot01$, $n = 2(2)50$ and $k = 2(1)11$. When $n_1 = n_2 = \ldots = n_{k-1} = n$, we can obtain the critical values c_α from the tables of Armitage and Krishnaiah (1964) for $\alpha = 0\cdot10, 0\cdot05, 0\cdot025, 0\cdot01$, $n = 1(1)19$, $k = 2(1)13$ and $n_k = 5(1)45$. The simultaneous confidence intervals associated with the above test procedure are given by

$$P\left[\frac{\sigma_k^2}{\sigma_i^2} \leq \frac{s_k^2}{s_i^2} c_{i\alpha}\,;\ i=1, 2, \ldots, k-1\right] = (1-\alpha) \tag{2.2}$$

where $c_{i\alpha}$'s are given by (2.1). We will now discuss about another one-sided version of the test considered in Gnanadesikan (1959).

Let $A^{**}_{2} = \bigcup A^{**}_{ik}$ where $A^{**}_{ik} : \sigma_i^2 < \sigma_k^2$. A procedure is proposed in Krishnaiah and Armitage (1964) to test $H_{1k}, H_{2k}, \ldots, H_{k-1,k}$ and H simultaneously against the respective alternatives $A^{**}_{1k}, A^{**}_{2k}, \ldots, A^{**}_{k-1,k}$ and A^{**}_{2}. When $n_1 = n_2 \ldots = n_k = n$, the 25%, 10%, 5% and 1% critical values which are useful in the application of this test procedure can be obtained from the tables of Gupta and Sobel (1962) for $n = 2(2)50$ and $k = 2(1)11$. The 10%, 5%, 2·5% and 1% critical values for this test procedure can be obtained from the tables of Krishnaiah and Armitage (1964) for $k = 2(1)13$, $n_k = 5(1)45$ and $m = 1(1)20$ when $n_1 = n_2 = \ldots = n_{k-1} = m$. The simultaneous confidence intervals associated with this test procedure are given by

$$P\left[\frac{\sigma_k^2}{\sigma_i^2} \geq \frac{s_k^2}{s_i^2} d_{i\alpha}\,;\ i=1, 2, \ldots, k-1\right] = (1-\alpha) \tag{2.3}$$

where $d_{i\alpha}$'s are chosen such that

$$P[F_{ik} \geq d_{i\alpha}\,;\ i=1, 2, \ldots, k-1 | H] = (1-\alpha). \tag{2.4}$$

We will now propose a test procedure for testing $H_{12}, H_{23}, \ldots, H_{k-1,k}$ and H simultaneously against the respective alternatives $A_{12}, A_{23}, \ldots, A_{k-1,k}$ and A_2. According to this procedure, we accept $H_{i,i+1}$ if

$$f_{i\alpha} \leq F_{i,i+1} \leq g_{i\alpha} \tag{2.5}$$

and reject $H_{i,i+1}$ otherwise. The constants $f_{i\alpha}$ and $g_{i\alpha}$ are chosen such that

$$\text{(2.6)} \qquad P[f_{i\alpha} \leq F_{i,i+1} \leq g_{i\alpha}\,;\ i=1, 2, \ldots, k-1 | H] = (1-\alpha).$$

The optimum choice of the critical values $f_{i\alpha}$ and $g_{i\alpha}$ is not known. For practical purposes, we impose the restriction that the acceptance regions (2.5) are of equal size. In addition, we impose the restriction that, for each i, the test with the acceptance region (2.5) is locally unbiased.

When H is true, $n_1 s_1^2, n_2 s_2^2, \ldots, n_k s_k^2$ are distributed independently as central chi-square variates with $n_1, n_2, \ldots, n_k$ degrees of freedom. Starting from the joint distribution of $s_1^2, s_2^2, \ldots, s_k^2$, making the transformations $F_{i,i+1} = s_i^2/s_{i+1}^2$, $s_k^2 = s_k^2$ and integrating out s_k^2 we obtain the following expression for the joint frequency function of $F_{12}, F_{23}, \ldots, F_{k-1,k}$ when H is true:

$$\text{(2.7)} \quad f(F_{12}, F_{23}, \ldots F_{k-1,k} | H) = \frac{(n_1/n_k)\left[\prod_{j=1}^{k-1} n_j \prod_{i=j}^{k-1} F_{i,i+1}/n_k\right]^{\frac{1}{2}(n_k-2)}}{\prod_{j=1}^{k} \Gamma[n_j/2]\left[1 + n_k^{-1} \sum_{j=1}^{k-1} n_j \prod_{i=j}^{k-1} F_{i,i+1}\right]^{\frac{1}{2}\Sigma n_j}}$$

The simultaneous confidence intervals associated with the above test are given by

$$\text{(2.8)} \quad P[f_{i\alpha} s_{i+1}^2/s_i^2 \leq \sigma_{i+1}^2/\sigma_i^2 \leq g_{i\alpha} s_{i+1}^2/s_i^2,\ i=1, 2, \ldots, k-1] = (1-\alpha).$$

The above simultaneous confidence intervals can be derived by using the fact that

$$\text{(2.9)} \quad \begin{aligned} &P[f_{i\alpha} \leq F_{i,i+1} \leq g_{i\alpha}\,;\ i=1, 2, \ldots, k-1 | H] \\ &= P[f_{i\alpha} \leq \sigma_{i+1}^2 F_{i,i+1}/\sigma_i^2 \leq g_{i\alpha}\,;\ i=1, 2, \ldots, k-1 | \cap A_i] = (1-\alpha). \end{aligned}$$

A procedure is proposed below for testing $H_{12}, H_{23}, \ldots, H_{k-1,k}$, and H simultaneously against the respective alternatives $A_{12}^{*}, A_{23}^{*}, \ldots, A_{k-1,k}^{*}$ and A_3^{*} where $A_3^{*} = \bigcup_{i=1}^{k-1} A_{i,i+1}^{*}$ and $A_{i,i+1}^{*} : \sigma_i^2 > \sigma_{i+1}^2$. According to this procedure, we accept or reject $H_{i,i+1}$ according as

$$F_{i,i+1} \lessgtr a_{i\alpha}$$

where $a_{i\alpha}$'s are chosen such that

$$\text{(2.10)} \qquad P[F_{i,i+1} \leq a_{i\alpha}\,;\ i=1, 2, \ldots, k-1 | H] = (1-\alpha).$$

The simultaneous confidence intervals associated with the above test are given by

$$\text{(2.11)} \qquad P[\sigma_{i+1}^2/\sigma_i^2 \leq a_{i\alpha} s_{i+1}^2/s_i^2\,;\ i=1, 2, \ldots, k-1] = (1-\alpha).$$

For practical purposes, we choose the critical values $a_{i\alpha}$ in (2.10) to be equal.

We test $H_{12}, H_{23}, \ldots, H_{k-1,k}$ and H simultaneously against the respective alternatives $A_{12}^{**}, A_{23}^{**}, \ldots, A_{k-1,k}^{**}$ and A_3^{**} where $A_3^{**} = \bigcup_{i=1}^{k-1} A_{i,i+1}^{**}$ and $A_{i,i+1}^{**} : \sigma_i^2 < \sigma_{i+1}^2$, by accepting or rejecting $H_{i,i+1}$ according as

$$F_{i,i+1} \gtrless b_{i\alpha}$$

where $b_{i\alpha}$'s are chosen such that

(2.12) $\quad P[F_{i,i+1} \geq b_{i\alpha}\,;\; i=1, 2, \ldots, k-1 | H] = (1-\alpha).$

The simultaneous confidence intervals associated with the above test are given by

$$(2.13) \qquad P\left[\frac{\sigma_{i+1}^2}{\sigma_i^2} \geq \frac{s_{i+1}^2}{s_i^2} b_{i\alpha}\,;\; i=1, 2, \ldots, k-1\right] = (1-\alpha).$$

As before we can, for practical purposes, choose the critical values $b_{i\alpha}$'s to be equal. When $k=3$, the values of α in (2.12) can be computed for given values of $b_{i\alpha}$ and n_i by using the method discussed in Bechhofer and Sobel (1954) and Bozivich, Bancroft and Hartley (1956).

3. General Remarks

Hartley's F_{max} is useful when the experimenter is interested in testing the hypotheses H_{rs} $(r \neq s = 1, 2, \ldots, k)$ simultaneously and the sample sizes are equal. But there are many situations when the experimenter is interested in testing a subset of these hypotheses. In these situations, it is not desirable to use the F_{max} test since alternative simultaneous test procedures yield shorter (in terms of lengths) confidence intervals. For example, if we are interested in testing $H_{12}, H_{23}, \ldots, H_{k-1,k}$ and H simultaneously against the respective alternatives $A_{12}, A_{23}, \ldots, A_{k-1,k}$ and A_3, the lengths of the confidence intervals associated with the simultaneous test procedure proposed in this paper are shorter than those associated with the F_{max} test if the critical values $f_{i\alpha}$ and $g_{i\alpha}$ in (2.6) are chosen such that $g_{i\alpha} = g_\alpha$ and $f_{i\alpha} = g_\alpha^{-1}$ for $i=1, 2, \ldots, k-1$ when the sample sizes are equal. Similarly, if we choose the critical values properly, the lengths of the confidence intervals associated with Gnanadesikan's test are shorter than the lengths of the corresponding confidence intervals associated with the F_{max} test.

Now let β denote the Type II error associated with the test proposed in this paper for testing H against A_2^*. When $\bigcap_{i=1}^{k-1} A_{ik}^*$ is true,

$$\begin{aligned} 1-\beta &= P[F_{ik} \geq c_{i\alpha}\,;\; i=1, 2, \ldots, k-1 | \bigcap_{i=1}^{k-1} A_{ik}^*] \\ &= \int_{c_{1,\alpha}/\delta_{1,k}}^{\infty} \cdots \int_{c_{k-1,\alpha}/\delta_{k-1,k}}^{\infty} f(F_{1,k}, \ldots, F_{k-1,k} | H) \prod_{i=1}^{k-1} dF_{i,k} \end{aligned}$$

where $\delta_{i,k} = \sigma_i^2/\sigma_k^2$. It is now evident that $(1-\beta)$ increases as each non-centrality parameter δ_{ik} increases. The above proof can be easily modified when only some of the A_{ik}^*'s are true. We can similarly show that other one-sided test procedures considered in this paper are monotonic increasing functions of the non-centrality parameters.

The test procedures discussed in the present paper are analogous to some special cases of the tests considered in Krishnaiah (1965*a*, 1965*b*) for multiple comparisons of means.

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